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# ON PARTITIONING THE ORBITALS OF A TRANSITIVE PERMUTATION GROUP

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ABSTRACT. Let $G$ be a permutation group on a set $\Omega$ with a transitive normal subgroup $M$. Then $G$ acts on the set $\mathrm{Orbl}(M,\Omega)$ of nontrivial $M$-orbitals in the natural way, and here we are interested in the case where $\mathrm{Orbl}(M,\Omega)$ has a partition $\mathcal{P}$ such that $G$ acts transitively on $\mathcal{P}$. The problem of characterising such tuples $(M,G,\Omega,\mathcal{P})$, called TODs, arises naturally in permutation group theory, and also occurs in number theory and combinatorics. The case where $|\mathcal{P}|$ is a prime-power is important in algebraic number theory in the study of arithmetically exceptional rational polynomials. The case where $|\mathcal{P}|=2$ exactly corresponds to self-complementary vertex-transitive graphs, while the general case corresponds to a type of isomorphic factorisation of complete graphs, called a homogeneous factorisation. Characterising homogeneous factorisations is an important problem in graph theory with applications to Ramsey theory. This paper develops a framework for the study of TODs, establishes some numerical relations between the parameters involved in TODs, gives some reduction results with respect to the $G$-actions on $\Omega$ and on $\mathcal{P}$, and gives some construction methods for TODs.

## 1. Introduction

Each transitive permutation group $M$ on a set $\Omega$ has a natural induced action on the set

$$\Omega^{(2)}=(\Omega\times\Omega)\setminus\{(\alpha,\alpha)\mid\alpha\in\Omega\}=\{(\alpha,\beta)\mid\alpha\neq\beta\in\Omega\},$$

given by $(\alpha,\beta)^x=(\alpha^x,\beta^x)$ for $\alpha,\beta\in\Omega$ with $\alpha\neq\beta$ and $x\in M$. The $M$-orbits in $\Omega^{(2)}$ are called *$M$-orbitals* on $\Omega$, and a partition $\mathcal{P}$ of $\Omega^{(2)}$ is called an *$M$-orbital decomposition* if each class $P\in\mathcal{P}$ is a union of one or more $M$-orbitals. Let $\mathrm{Orbl}(M,\Omega)$ denote the set of $M$-orbits in $\Omega^{(2)}$. In this paper we investigate the situation where $M<G\leq\mathrm{Sym}(\Omega)$, and $G$ induces a transitive action on an $M$-orbital decomposition $\mathcal{P}$ of $\Omega^{(2)}$; that is to say, we assume

(i) for all $P\in\mathcal{P}$ and $g\in G$, $P^g\in\mathcal{P}$,
(ii) for $P,P'\in\mathcal{P}$, there exists $g\in G$ such that $P^g=P'$, and
(iii) $\mathcal{P}$ is refined by $\mathrm{Orbl}(M,\Omega)$.

If these conditions hold, then we call the tuple $(M,G,\Omega,\mathcal{P})$ a *transitive orbital decomposition*, or a *TOD* for short. The cardinalities $|\Omega|$ and $|\mathcal{P}|$ are called the *degree* and the *index* of the TOD, and sometimes we refer to $(M,G,\Omega,\mathcal{P})$ as a $k$-TOD if $k=|\mathcal{P}|$.

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Interest in transitive orbital decompositions has arisen from several different areas. The case $k = 2$ corresponds to self-complementary graphs, which are discussed in more detail below. The case where $k$ is a prime-power arose also in algebraic number theory in the study of arithmetically exceptional rational polynomials [11]. (See [7] for the more general context of these investigations.)

Assume that $k = 2$. Then the two classes $P, P'$ in $\mathcal{P}$ can be regarded as the edge sets of directed graphs (digraphs) $\Gamma = (\Omega, P)$ and $\Gamma' = (\Omega, P')$ such that the group $M$ is a vertex-transitive automorphism group of both $\Gamma$ and $\Gamma'$, and each $g \in G$ that interchanges $P$ and $P'$ induces a graph isomorphism between $\Gamma$ and $\Gamma'$. Thus $\Gamma$, $\Gamma'$ is a pair of self-complementary vertex-transitive digraphs. Moreover, if $\mathcal{P}$ is *symmetric*, in the sense that each $P \in \mathcal{P}$ is equal to its reverse $P^* = \{(\beta, \alpha) \mid (\alpha, \beta) \in P\}$, and if $\mathcal{P}$ has index $k = 2$, then the two graphs $\Gamma, \Gamma'$ may be considered as undirected self-complementary graphs, namely, $\Gamma = (\Omega, E)$ and $\Gamma' = (\Omega, E')$, where the edge sets $E, E'$ are the sets of unordered pairs $\{\alpha, \beta\}$ from $\Omega$ with $(\alpha, \beta)$ in $P, P'$, respectively.

The study of vertex-transitive self-complementary graphs began with a construction of a family of self-complementary circulant graphs given by H. Sachs [24]. Vertex-transitive self-complementary graphs have received considerable attention in the literature, see for example [14, 17, 18, 20, 25, 27], and they have been used to investigate Ramsey numbers [3, 4, 5]. Most of the known vertex-transitive self-complementary graphs are Cayley graphs, see for example [14, 18, 25, 19, 23]; the first infinite family of vertex-transitive self-complementary graphs that are not Cayley graphs was constructed recently in [17]. With regard to TODs of arbitrary index $k$, we give necessary and sufficient conditions for the existence of a $k$-TOD in Proposition 3.3, and the proof of this result includes a general construction for them. Explicit constructions are given in Section 6.

A lot of effort was expended in determining the positive integers $n$ such that there exist vertex-transitive self-complementary graphs with $n$ vertices, see [1, 8, 15, 20, 27], and recently, Muzychuk [20] completely determined such positive integers. One of the major results of this paper, Theorem 1.1, is a generalisation of Muzychuk's result. A $k$-TOD $(M, G, \Omega, \mathcal{P})$ is called *cyclic* if the transitive permutation group $G^{\mathcal{P}}$ induced by $G$ on $\mathcal{P}$ is cyclic. Thus a pair of vertex-transitive self-complementary digraphs corresponds to a cyclic 2-TOD. Our result classifies the possibilities for the degree of a cyclic $k$-TOD, and the case $k = 2$ is the result of Muzychuk [20].

**Theorem 1.1.** *Let $k$ be an integer such that $k > 1$, and let $n = r_1^{d_1} r_2^{d_2} \dots r_m^{d_m}$, where the $r_i$ are distinct primes, $d_i \geq 1$, and $m \geq 1$. Then*

(i) *there exists a cyclic $k$-TOD of degree $n$ if and only if, for all $i = 1, \dots, m$,*
$$r_i^{d_i} \equiv 1 \pmod{k};$$

(ii) *there exists a cyclic symmetric $k$-TOD of degree $n$ if and only if, for all $i = 1, 2, \dots, m$,*
$$r_i^{d_i} \equiv 1 \pmod{2k} \text{ if } r_i \text{ is odd, or}$$
$$r_i^{d_i} \equiv 1 \pmod{k} \text{ if } r_i = 2.$$

For a transitive group $M \leq \mathrm{Sym}(\Omega)$, let $M^{(2)}$ denote the 2-closure of $M$ in the sense of Wielandt; that is, $M^{(2)}$ is the largest subgroup of $\mathrm{Sym}(\Omega)$ with the same orbits as $M$ in $\Omega^{(2)}$. One way that a subgroup $G \leq \mathrm{Sym}(\Omega)$ may leave invariant a partition refined by $\mathrm{Orbl}(M, \Omega)$ is if $G$ permutes the $M$-orbitals setwise. We show

(Proposition 2.1) that a subgroup $G \leq \mathrm{Sym}(\Omega)$ leaves the set $\mathrm{Orbl}(M, \Omega)$ invariant (that is, $\Delta^g \in \mathrm{Orbl}(M, \Omega)$ for all $\Delta \in \mathrm{Orbl}(M, \Omega)$ and all $g \in G$) if and only if $G$ normalises $M^{(2)}$. However, whether or not $G$ leaves $\mathrm{Orbl}(M, \Omega)$ invariant in a TOD $(M, G, \Omega, \mathcal{P})$, we may always replace $M$ by the kernel $\hat{M}$ of the action of $G$ on $\mathcal{P}$, and thereby obtain a TOD $(\hat{M}, G, \Omega, \mathcal{P})$ with $\hat{M}$ normal in $G$. From Subsection 2.2 onwards we shall always assume that $M$ is normalised by $G$. We begin by presenting in Section 2 several elementary properties of TODs and in Section 3 constructions of new TODs from a given TOD. Some of these constructions yield examples of cyclic TODs.

If $(M, G, \Omega, \mathcal{P})$ is a $k$-TOD and $\mathcal{B}$ is a nontrivial block system for $G$ in $\Omega$, then the induced structure $(M_B^B, G_B^B, B, \mathcal{P}_B)$ on a block $B \in \mathcal{B}$ is also a $k$-TOD (Lemma 4.1), but there seems to be no natural induced TOD corresponding to the actions of $M$ and $G$ on $\mathcal{B}$ (see Subsection 4.2). However, for cyclic $k$-TODs, we are able to induce a cyclic $k$-TOD for the quotient action on $\mathcal{B}$.

**Theorem 1.2.** *Let $(M, G, \Omega, \mathcal{P})$ be a cyclic $k$-TOD. Then for any $G$-invariant partition $\mathcal{B}$ of $\Omega$, there exists a partition $\mathcal{Q}$ of $\mathcal{B}^{(2)}$ such that $(M^{\mathcal{B}}, G^{\mathcal{B}}, \mathcal{B}, \mathcal{Q})$ is a cyclic $k$-TOD.*

This theorem will be proved in Section 3. It raises the question of classifying cyclic $k$-TODs $(M, G, \Omega, \mathcal{P})$ with $G$ primitive on $\Omega$. For the case where $k$ is a prime-power, a classification is given in [10]. In Section 4 we give several constructions of cyclic TODs proving the existence assertions of Theorem 1.1, and we complete the proof of Theorem 1.1 in Section 5.

1.1. **Further graph-theoretic links.** From another viewpoint, a symmetric TOD of degree $n$ is a special type of isomorphic factorisation of the complete graph $K_n$ on $n$ vertices. An *isomorphic factorisation* of $K_n$ with vertex set $V$ is a decomposition $\{E_1, \ldots, E_k\}$ of the unordered pairs of vertices such that the $k$ graphs $(V, E_1), \ldots, (V, E_k)$ are pairwise isomorphic; the graphs $(V, E_i)$ are called the *factors* of the factorisation. Isomorphic factorisations of complete graphs have been investigated for a long time, see for instance [12, 13]. If $(M, G, \Omega, \mathcal{P})$ is a symmetric $k$-TOD of degree $n$ and if $E_i = \{\{x, y\} \mid (x, y) \in P_i\}$ where $\mathcal{P} = \{P_1, \ldots, P_k\}$, then $\{E_1, \ldots, E_n\}$ is an isomorphic factorisation of $K_n$ with vertex set $\Omega$ with the additional property that the group $G$ permutes $\{E_1, \ldots, E_k\}$ transitively. Thus, isomorphisms between each pair $(V, E_i)$ and $(V, E_j)$ can be induced by elements of $G$. Isomorphic factorisations of $K_n$ with this property are called *homogeneous factorisations.*

We end this section with a discussion of two special classes of TODs. Let $(M, G, \Omega, \mathcal{P})$ be a $k$-TOD such that $\mathcal{P} = \mathrm{Orbl}(M, \Omega)$, that is, $\mathcal{P}$ is a trivial partition of $\mathrm{Orbl}(M, \Omega)$. It then follows that $G$ is a 2-transitive permutation group on $\Omega$, and $M$ is a permutation group of rank $k + 1$. By Proposition 2.1, we may assume that $G$ contains $M^{(2)}$, and then $M^{(2)}$ is a normal subgroup of $G$ of rank $k + 1$. Inspecting the classification of 2-transitive permutation groups, see [2], we see that either $G$ is affine, or $M \cong \mathrm{PSL}_2(8)$ and $G \cong \mathrm{Aut}\,(\mathrm{PSL}_2(8))$. The latter indeed gives rise to a 3-TOD $(M, G, \Omega, \mathcal{P})$ such that $\mathcal{P} = \mathrm{Orbl}(M, \Omega)$, which is symmetric and has degree 28. This, in particular, shows that the complete graph $K_{28}$ may be factorised into three isomorphic arc-transitive graphs of valency 9. Further, this, together with Theorem 1.1, also shows that a complete graph $K_n$ having a nontrivial homogeneous factorisation with $M$-arc-transitive factors implies that $n$ is a prime-power or $n = 28$. The special case where $k = 2$ corresponds to arc-transitive self-complementary graphs, which are classified in [21, 28].

A more general interesting class of TODs is the class of homogeneous factorisations of complete graphs with edge-transitive factors. This corresponds to the class of symmetric $k$-TODs $(M, G, \Omega, \mathcal{P})$ such that each $P_i \in \mathcal{P}$ is an orbital, or a union of two paired orbitals of $M^\Omega$. In this case, it follows that we can take $G$ to be a 2-homogeneous permutation group on $\Omega$ with a transitive normal subgroup $M^{(2)}$ of rank $k+1$. The TOD arising from $\mathrm{PSL}_2(8)$ is also an example for this case.

In subsequent work [16], a complete description will be given of homogeneous factorisations of complete graphs with arc-transitive or edge-transitive factors.

## 2. General properties of TODs

2.1. **On the normality of $M$.** Some TODs $(M, G, \Omega, \mathcal{P})$ arise with the group $G$ having an induced action on the set $\mathrm{Orbl}(M, \Omega)$ of $M$-orbitals. This condition is not part of the definition of a TOD. However, we do not have any examples where it does not hold. We first prove the result mentioned in the introduction, which relates this property to the 2-closure of $M$.

**Proposition 2.1.** *Let $M$ be a transitive permutation group on a set $\Omega$. Then a subgroup $G \le \mathrm{Sym}(\Omega)$ leaves $\mathrm{Orbl}(M, \Omega)$ invariant if and only if $G$ normalises $M^{(2)}$.*

*Proof.* Suppose that $G$ normalises $M^{(2)}$. Let $g \in G$ and $\Delta \in \mathrm{Orbl}(M, \Omega)$. We need to prove that $\Delta^g \in \mathrm{Orbl}(M, \Omega)$. For $x \in M$, we have $x^{g^{-1}} \in M^{(2)}$, and hence $\Delta^{gxg^{-1}} = \Delta^{x^{g^{-1}}} = \Delta$. So $\Delta^{gx} = \Delta^g$, that is, $\Delta^g$ is $M$-invariant. For any $(\alpha, \beta), (\gamma, \delta) \in \Delta$, there exists $x \in M$ such that $(\alpha, \beta)^x = (\gamma, \delta)$. Now $g^{-1}xg \in M^{(2)}$, $(\alpha^g, \beta^g), (\gamma^g, \delta^g) \in \Delta^g$, and $(\alpha^g, \beta^g)^{g^{-1}xg} = (\gamma^g, \delta^g)$. It follows that $M^{(2)}$ is transitive on $\Delta^g$, and so $\Delta^g$ is an $M^{(2)}$-orbital. Thus by the definition of $M^{(2)}$, $\Delta^g \in \mathrm{Orbl}(M, \Omega)$, and hence $G$ leaves $\mathrm{Orbl}(M, \Omega)$ invariant.

Conversely, suppose that $G$ leaves $\mathrm{Orbl}(M, \Omega)$ invariant. Let $x \in M^{(2)}$ and $g \in G$. We claim that $x^g \in M^{(2)}$. Take an arbitrary element $\Delta \in \mathrm{Orbl}(M, \Omega)$. Then $\Delta^{g^{-1}} \in \mathrm{Orbl}(M, \Omega)$, and hence $(\Delta^{g^{-1}})^x = \Delta^{g^{-1}}$. Therefore, $\Delta^{x^g} = \Delta^{g^{-1}xg} = \Delta$, and so $x^g$ fixes every element of $\mathrm{Orbl}(M, \Omega)$. Thus $x^g \in M^{(2)}$, and so $G$ normalises $M^{(2)}$. □

If $(M, G, \Omega, \mathcal{P})$ is a $k$-TOD and $G$ normalises $M^{(2)}$, then $(M^{(2)} \cap G, G, \Omega, \mathcal{P})$ is also a $k$-TOD, and by Lemma 2.1, $M^{(2)} \cap G$ is a normal subgroup of $G$. Whether or not this is the case, if $\hat{M}$ is the kernel of the action of $G$ on $\mathcal{P}$, then $(\hat{M}, G, \Omega, \mathcal{P})$ is a $k$-TOD. *We will assume from now on that, for a $k$-TOD $(M, G, \Omega, \mathcal{P})$, the group $M$ is a normal subgroup of $G$.* Moreover, we note the following.

**Lemma 2.2.** *Let $(M, G, \Omega, \mathcal{P})$ be a $k$-TOD, and let $K$ be the kernel of the $G$-action on $\mathcal{P}$. Assume that $N$ is a normal subgroup of $G$ such that $N \le K$ and $N$ is transitive on $\Omega$. Then $(N, G, \Omega, \mathcal{P})$ is a $k$-TOD.*

*Proof.* This is clear from the definition of TODs. □

2.2. **TODs and partitions of $\Omega$.** Choose a point $\omega \in \Omega$. Now we give a relation between a partition of $\Omega^{(2)}$ and a partition of $\Omega \setminus \{\omega\}$. For a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ of $\Omega^{(2)}$, let $P_i(\omega) = \{\omega' \in \Omega \mid (\omega, \omega') \in P_i\}$, and let

$\mathcal{P}(\omega) = \{P_1(\omega), P_2(\omega), \ldots, P_k(\omega)\}$. Let $M$ be a transitive permutation group on $\Omega$, and let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ be a partition of $\Omega^{(2)}$ refined by $\mathrm{Orbl}(M, \Omega)$. Then $\mathcal{P}(\omega)$ is a partition of $\Omega \setminus \{\omega\}$ refined by the set of $M_\omega$-orbits in $\Omega \setminus \{\omega\}$. Conversely, if $\mathcal{P}(\omega)$ is a partition of $\Omega \setminus \{\omega\}$ refined by the set of $M_\omega$-orbits in $\Omega \setminus \{\omega\}$, then we obtain a partition $\mathcal{P}$ of $\Omega^{(2)}$ refined by $\mathrm{Orbl}(M, \Omega)$, by defining $\mathcal{P} = \{P_1, \ldots, P_k\}$ with $P_i = \{(\omega, \omega')^g \in \Omega^{(2)} \mid \omega' \in P_i(\omega), g \in M\}$. The next lemma shows that for an overgroup $G$ of $M$ that leaves $\mathcal{P}$ invariant, the $G$-action on $\mathcal{P}$ is equivalent to the $G_\omega$-action on $\mathcal{P}(\omega)$. Whenever a group $G$ has an action on a set $\mathcal{P}$ we shall denote by $G^{\mathcal{P}}$ the permutation group on $\mathcal{P}$ induced by $G$.

**Lemma 2.3.** *Let $M$ be a transitive permutation group on $\Omega$, and suppose that $\mathcal{P}$ is a partition of $\Omega^{(2)}$ refined by $\mathrm{Orbl}(M, \Omega)$. Let $G$ be such that $M \lhd G \leq \mathrm{Sym}(\Omega)$, and let $\omega \in \Omega$. Then*

1. *$\mathcal{P}$ is $G$-invariant if and only if $\mathcal{P}(\omega)$ is $G_\omega$-invariant;*
2. *in the case where $\mathcal{P}$ is $G$-invariant, $G^{\mathcal{P}} = G_\omega^{\mathcal{P}}$ and the $G_\omega$-actions on $\mathcal{P}$ and $\mathcal{P}(\omega)$ are equivalent; in particular, $G$ is transitive on $\mathcal{P}$ if and only if $G_\omega$ is transitive on $\mathcal{P}(\omega)$, and thus $(M, G, \Omega, \mathcal{P})$ is a TOD if and only if $G_\omega$ is transitive on $\mathcal{P}(\omega)$.*

*Proof.* Since $M$ is transitive on $\Omega$, $G = MG_\omega$, and thus each element $x \in G$ may be written as $x = gy$ for some $g \in M$ and some $y \in G_\omega$. Then for $P_i, P_j \in \mathcal{P}$, $P_i^x = P_i^y = P_j$ if and only if $P_i(\omega)^y = P_j(\omega)$. Thus $\mathcal{P}$ is $G$-invariant if and only if $\mathcal{P}(\omega)$ is $G_\omega$-invariant. Moreover, since $G = MG_\omega$, we have $G^{\mathcal{P}} = G_\omega^{\mathcal{P}}$ and the $G_\omega$-action on $\mathcal{P}$ is equivalent to the $G_\omega$-action on $\mathcal{P}(\omega)$. In particular, $G$ is transitive on $\mathcal{P}$ if and only if $G_\omega$ is transitive on $\mathcal{P}(\omega)$. □

2.3. **A congruence involving $n$ and $k$.** Lemma 2.3 implies a congruence relation that must be satisfied by the parameters $n$ and $k$ for $k$-TODs of degree $n$.

**Lemma 2.4.** *Let $(M, G, \Omega, \mathcal{P})$ be a $k$-TOD, where $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$. Then for each $G_\omega$-orbit $\Delta$ in $\Omega \setminus \{\omega\}$, $\{\Delta \cap P_i(\omega) \mid 1 \leq i \leq k\}$ is a $G_\omega$-invariant partition of $\Delta$; in particular, $k$ divides $|\Delta|$.*

*Proof.* By definition, $\mathcal{P}$ is a $G$-invariant partition of $\Omega^{(2)}$. Thus by Lemma 2.3, $\mathcal{P}(\omega)$ is a $G_\omega$-invariant partition of $\Omega \setminus \{\omega\}$. It follows, since $G_\omega$ fixes $\Delta$ setwise, that $\{\Delta \cap P_i(\omega) \mid 1 \leq i \leq k\}$ is a $G_\omega$-invariant partition of $\Delta$. Also, by Lemma 2.3, $G_\omega$ is transitive on $\mathcal{P}(\omega)$ and hence also on $\{\Delta \cap P_i(\omega) \mid 1 \leq i \leq k\}$. Thus the sets $\Delta \cap P_i(\omega)$ all have the same size; so $|\Delta|$ is divisible by $k$. □

**Lemma 2.5.** *For positive integers $n$ and $k$, if there exists a $k$-TOD of degree $n$, then $n \equiv 1 \pmod{k}$, and in particular, $k$ is coprime to $n$. Moreover, if the $k$-TOD is symmetric and $n$ is odd, then $n \equiv 1 \pmod{2k}$.*

*Proof.* Let $(M, G, \Omega, \mathcal{P})$ be a $k$-TOD of degree $n$, and let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$. By Lemma 2.4, it follows that $k$ divides the length of each $G_\omega$-orbit in $\Omega \setminus \{\omega\}$, and hence $k$ divides $n - 1$.

Now assume that $\mathcal{P}$ is symmetric. Then each $P_i$ is a symmetric relation on $\Omega$, and so there are $n(n-1)/(2k)$ unordered pairs $\{\alpha, \beta\}$ such that $P_i$ contains $(\alpha, \beta)$ and $(\beta, \alpha)$. If $n$ is odd, then $2k$ is coprime to $n$ since $k$ is coprime to $n$, and thus $(n-1)/(2k)$ is an integer, that is, $n \equiv 1 \pmod{2k}$. □

## 3. Some construction methods for TODs

3.1. **TODs and partitions of** $\mathrm{Orbl}(M,\Omega)$**.** Suppose that $(M,G,\Omega,\mathcal{P})$ is a $k$-TOD with $M \lhd G$. Then, for each $G$-orbit $Q$ in $\mathrm{Orbl}(M,\Omega)$, the partition $\mathcal{P}$ of $\Omega^{(2)}$ determines a partition $\mathcal{B}(Q) = \{B(P) \mid P \in \mathcal{P}\}$ of $Q$, where $B(P)$ is the set of $M$-orbitals $\Delta \in Q$ such that $\Delta \subseteq P$. The next result explores the connection between the existence of a $k$-TOD and the existence of $k$-part partitions $\mathcal{B}(Q)$ for the $G$-orbits $Q$ in $\mathrm{Orbl}(M,\Omega)$. This result underlies our later constructions of cyclic TODs in Section 6.

Actions of a group $G$ on $\Omega$ and $\Omega'$ are said to be *permutationally equivalent* if there is a bijection $f\colon \Omega \to \Omega'$ such that, for all $g \in G$ and $\omega \in \Omega$, $(\omega^g)f = (\omega f)^g$. For a transitive subgroup $M \leq \mathrm{Sym}(\Omega)$ and a subset $X \subseteq \mathrm{Orbl}(M,\Omega)$, define $X^* = \{\Delta^* \mid \Delta \in X\}$. We say that $X$ is *symmetric* if $X = X^*$.

**Construction 3.1.** Let $M$ be a transitive permutation group on $\Omega$, and let $M \trianglelefteq G \leq \mathrm{Sym}(\Omega)$. Suppose that for each $G$-orbit $Q$ in $\mathrm{Orbl}(M,\Omega)$, there exists a $G$-invariant partition $\mathcal{B}(Q)$ of $Q$ with $k$ parts. Choose a particular $G$-orbit $Q$ and let $B \in \mathcal{B}(Q)$. Suppose that the actions induced by $G$ on the $\mathcal{B}(Q)$ are permutationally equivalent, that is, for any other $G$-orbit $Q'$ in $\mathrm{Orbl}(M,\Omega)$, there exists a bijection $f_{Q'}\colon \mathcal{B}(Q) \to \mathcal{B}(Q')$ such that, for all $C \in \mathcal{B}(Q)$ and all $g \in G$, $(C^g)f_{Q'} = (Cf_{Q'})^g$. Let $J = \{g_1,\dots,g_k\}$ be a set of coset representatives for the setwise stabiliser $G_B$ in $G$, and note that $G_B = G_{(B)f_{Q'}}$ for each $Q'$. Define $P_1$ to be the union, over all $G$-orbits $Q'$ in $\mathrm{Orbl}(M,\Omega)$, of all ordered pairs of points contained in $M$-orbitals in $(B)f_{Q'}$; that is,

$$P_1 = \bigcup_{Q'} \Big( \bigcup_{\Delta \in (B)f_{Q'}} \Delta \Big).$$

Let $P_i = P_1^{g_i}$ for $i = 1,\dots,k$, and let $\mathcal{P} = \{P_1,\dots,P_k\}$.

That the 4-tuple $(M,G,\Omega,\mathcal{P})$ produced by this construction is a TOD is proved below.

**Lemma 3.2.** *For $M,G,\mathcal{P}$ as in Construction 3.1, $(M,G,\Omega,\mathcal{P})$ is a $k$-TOD, and moreover it is symmetric if and only if*

(a) *for each symmetric orbit $R$, each part of the partition $\mathcal{B}(R)$ is symmetric, and*
(b) *for each nonsymmetric orbit $R$, and each $B \in \mathcal{B}(Q)$, we have $(Bf_R)^* = (B)f_{R^*}$.*

*Proof.* Suppose that $(M,G,\Omega,\mathcal{P})$ is a tuple produced in Construction 3.1. Since $G_B = G_{(B)f_{Q'}}$, it follows that $\mathcal{B}(Q') = \{(Bf_{Q'})^{g_i} \mid i = 1,\dots,k\}$ for each $Q'$, and therefore $\mathcal{P}$ is a partition of $\Omega^{(2)}$. Moreover,

$$P_i = \bigcup_{Q'} \Big( \bigcup_{\Delta \in Bf_{Q'}} \Delta \Big)^{g_i} = \bigcup_{Q'} \Big( \bigcup_{\Delta' \in (Bf_{Q'})^{g_i}} \Delta' \Big) = \bigcup_{Q'} \Big( \bigcup_{\Delta' \in (B^{g_i})f_{Q'}} \Delta' \Big).$$

Let $g \in G$. Then a similar argument gives $P_i^g = \bigcup_{Q'} (\bigcup_{\Delta \in (B^{g_i g})f_{Q'}} \Delta) = P_j$, where $B^{g_i g} = B^{g_j}$. Thus $\mathcal{P}$ is $G$-invariant, and $G^{\mathcal{P}}$ is permutationally equivalent to $G^{\mathcal{B}(Q)}$ and, in particular, is transitive. Thus $(M,G,\Omega,\mathcal{P})$ is a $k$-TOD.

Suppose that $(M,G,\Omega,\mathcal{P})$ is a symmetric $k$-TOD, and let $R$ be a symmetric orbit and $B \in \mathcal{B}(R)$. Let $\Delta \in B$, and suppose that $\Delta \subset P$ with $P \in \mathcal{P}$. Since $R$ is symmetric we have $\Delta^* \in R$, and since $(M,G,\Omega,\mathcal{P})$ is symmetric we have $\Delta^* \subseteq P$.

By Construction 3.1, $B$ consists of all orbitals $\Delta'$ such that $\Delta' \in R$ and $\Delta' \subseteq P$. Therefore $\Delta^* \in B$. It follows that $B = B^*$ is symmetric. Let $R$ be a nonsymmetric orbit. Let $\Delta \in R$ lie in the block $Bf_R$ of $\mathcal{B}(R)$. Then $\Delta^* \in (Bf_R)^*$ and $\Delta^* \in R^*$. Suppose that $\Delta \in P$, $P \in \mathcal{P}$. Since $\mathcal{P}$ is symmetric, we have $\Delta^* \in P$. By the definition of $f_{R^*}$, $(B)f_{R^*}$ is the set of orbitals $\Delta^* \in R^*$ that are contained in $P$. Hence $\Delta^* \in (B)f_{R^*}$, and so $(Bf_R)^* = (B)f_{R^*}$.

Conversely, suppose that, for each symmetric orbit $R$, each part of the partition $\mathcal{B}(R)$ is symmetric and for each nonsymmetric orbit $R$ and $C \in \mathcal{B}(Q)$, $(C)f_{R^*} = (Cf_R)^*$. We claim that $(M, G, \Omega, \mathcal{P})$ is symmetric. Since $G^{\mathcal{P}}$ is transitive, it is sufficient to prove that $P_1 = P_1^*$. Let $\Delta \subseteq P_1$ lie in an orbit $R$. If $R$ is symmetric, then $Bf_R$ is the set of orbitals $\Delta' \in R$ such that $\Delta' \in P_1$. Hence $\Delta \in Bf_R$. Now $\Delta^* \in R$ since $R$ is symmetric, and $\Delta^* \in Bf_R$ since the part $Bf_R$ of $\mathcal{B}(R)$ is symmetric, and hence $\Delta^* \subseteq P_1$. Now let $R$ be not symmetric. Again $\Delta \in Bf_R$; so $\Delta^* \in (Bf_R)^* = (B)f_{R^*}$ and, by Construction 3.1, $\Delta^* \subseteq P_1$. Thus $P_1 = P_1^*$; so $(M, G, \Omega, \mathcal{P})$ is symmetric. □

Now we obtain a set of necessary and sufficient conditions for the existence of TODs based on the action on $\mathrm{Orbl}(M, \Omega)$.

**Proposition 3.3.** (i) *Let $M$ be a transitive permutation group on $\Omega$, and let $M \trianglelefteq G \leq \mathrm{Sym}(\Omega)$. Then there exists a partition $\mathcal{P}$ of $\Omega^{(2)}$ such that $(M, G, \Omega, \mathcal{P})$ is a $k$-TOD if and only if for each $G$-orbit $Q$ in* $\mathrm{Orbl}(M, \Omega)$*, there exists a $G$-invariant partition $\mathcal{B}(Q)$ of $Q$ with $k$ parts, and the actions of $G$ on $\mathcal{B}(Q)$, for all $G$-orbits $Q$ in* $\mathrm{Orbl}(M, \Omega)$*, are pairwise permutationally equivalent.*

(ii) *Moreover, there exists a symmetric $k$-TOD $(M, G, \Omega, \mathcal{P})$ if and only if, in addition, for each symmetric $G$-orbit $Q$ in* $\mathrm{Orbl}(M, \Omega)$ *(if any such exists), there exists a partition $\mathcal{B}(Q)$ as in part* (i)*, each part of which is symmetric.*

*Proof.* Suppose that there exists $\mathcal{P} = \{P_1, \dots, P_k\}$ such that $(M, G, \Omega, \mathcal{P})$ is a $k$-TOD. Let $J = \{g_1, \dots, g_k\} \subset G$ be such that $P_1^{g_i} = P_i$ for each $i$. Since $G^{\mathcal{P}}$ is transitive, each $G$-orbit $Q$ in $\mathrm{Orbl}(M, \Omega)$ contains (at least one) $M$-orbital $\Delta \subseteq P_1$. Let $B_1$ be the set of $M$-orbitals $\Delta$ such that $\Delta \in Q$ and $\Delta \subseteq P_1$. For each $i$, set $B_i = B_1^{g_i}$. Then $B_i$ is the set of $M$-orbitals $\Delta$ such that $\Delta \in Q$ and $\Delta \subseteq P_i$. Thus $\mathcal{B}(Q) = \{B_1, \dots, B_k\}$ is a $G$-invariant partition of $Q$ with $k$ parts, and $G^{\mathcal{B}(Q)}$ is permutationally isomorphic to $G^{\mathcal{P}}$. Conversely, if suitable $G$-invariant partitions $\mathcal{B}(Q)$ exist for each $Q$, then Construction 3.1 gives the required $k$-TOD by Lemma 3.2. Thus part (i) is proved.

Suppose further that $(M, G, \Omega, \mathcal{P})$ is symmetric, and suppose that $Q$ is a symmetric $G$-orbit in $\mathrm{Orbl}(M, \Omega)$. Then, since $P_1 = P_1^*$, the set $B_1$ of $M$-orbitals $\Delta \in Q$ that are contained in $P_1$ satisfies $B_1^* = B_1$. By the definition of $\mathcal{B}(Q)$ above, each part of $\mathcal{B}(Q)$ is symmetric.

Conversely, suppose that for each symmetric orbit $Q$, there is a $\mathcal{B}(Q)$ with all parts symmetric. For any such $Q$, we choose a partition $\mathcal{B}(Q)$ with this extra property. Choose a particular $G$-orbit $Q$ in $\mathrm{Orbl}(M, \Omega)$, and for each $G$-orbit $R$ let $f_R$: $\mathcal{B}(Q) \to \mathcal{B}(R)$ be the bijection defining the permutational equivalence of the $G$-actions (taking $f_Q$ to be the identity map). Suppose that $R \neq R^*$. Then $R^*$ is also a $G$-orbit in $\mathrm{Orbl}(M, \Omega)$; so, in particular, $R \cap R^* = \emptyset$. If necessary we replace $\mathcal{B}(R^*)$ by $\mathcal{B}^*(R) := \{B^* \mid B \in \mathcal{B}(R)\}$, and we replace $f_{R^*}$ by $f_R^*$: $\mathcal{B}^*(Q) \to \mathcal{B}^*(R)$ defined by $(B)f_R^* = (Bf_R)^*$, for $B \in \mathcal{B}(Q)$. Let $(M, G, \Omega, \mathcal{P})$ be as in Construction 3.1 using these partitions $\mathcal{B}(R)$. Then by Lemma 3.2, $(G, M, \Omega, \mathcal{P})$ is a symmetric TOD. □

3.2. **Some TODs derived from a given one.** Our first construction varies the partition $\mathcal{P}'$ but involves the same subgroup $M$.

**Lemma 3.4.** *Let $(M, G, \Omega, \mathcal{P})$ be a $k$-TOD.*

(1) *If $\mathcal{P}'$ is a nontrivial $G$-invariant partition of* $\mathrm{Orbl}(M, \Omega)$ *refined by $\mathcal{P}$, then $k' = |\mathcal{P}'| \geq 2$, $k'$ divides $k$, and $(M, G, \Omega, \mathcal{P}')$ is a $k'$-TOD.*

(2) *If $H < G$ is such that $H^{\mathcal{P}}$ is semiregular and nontrivial with orbits of length $k'$, and $H$ normalises $M$, then $k'|k$, $k' \geq 2$, and $(M, \langle M, H\rangle, \Omega, \mathcal{P}')$ is a $k'$-TOD for some $H$-invariant partition $\mathcal{P}'$ refined by $\mathcal{P}$.*

*Proof.* (1). Since $G$ is transitive on $\mathcal{P}$ and $\mathcal{P}'$ is refined by $\mathcal{P}$, $G$ is transitive on $\mathcal{P}'$, and hence $(M, G, \Omega, \mathcal{P}')$ is a TOD of index $k'$ dividing $k$.

(2). Choose a representative from each $H$-orbit in $\mathcal{P}$, and let $P_1'$ be the union of these representatives. Set $\mathcal{P}' = \{(P_1')^h \mid h \in H\}$. Since $H$ is semiregular on $\mathcal{P}$ with orbits of length $k'$, it follows that $\mathcal{P}'$ is an $H$-invariant partition of $\Omega^{(2)}$ with $k'$ parts and refined by $\mathcal{P}$. □

The next construction is the key both to a reduction to consideration of TODs $(M, G, \Omega, \mathcal{P})$ with $G$ primitive on $\Omega$, and also to the proof of Theorem 1.1. For a partition $\mathcal{P}$ of $\Omega^{(2)}$ and a subset $\Delta \subset \Omega$, by the *restriction* of $\mathcal{P}$ to $\Delta^{(2)}$ we mean the partition $\mathcal{Q} = \{P_i \cap (\Delta \times \Delta) \mid 1 \leq i \leq k\}$ of $\Delta^{(2)}$. Let $M \leq G \leq \mathrm{Sym}(\Omega)$ with $M$ transitive. For $\Delta \in \mathrm{Orbl}(M, \Omega)$, the paired orbital $\Delta^* = \{(\beta, \alpha) \mid (\alpha, \beta) \in \Delta\}$ also lies in $\mathrm{Orbl}(M, \Omega)\}$. If $G$ leaves $\mathrm{Orbl}(M, \Omega)$ invariant, then, for each $g \in G$, $(\Delta^*)^g = (\Delta^g)^*$. For a subgroup $N \leq \mathrm{Sym}(\Omega)$ and $\omega \in \Omega$, we denote by $\omega^N$ the $N$-orbit containing $\omega$.

**Lemma 3.5.** *Let $(M, G, \Omega, \mathcal{P})$ be a $k$-TOD with $M$ normal in $G$, let $\omega \in \Omega$, and suppose that $N$ is a subgroup of $M$ with no fixed points in $\Omega$. Assume that $E \leq G_\omega$ is such that $E^{\mathcal{P}}$ is transitive and $E \leq \mathbf{N}_G(N)$. Set $F = NE$ and $\Delta = \omega^N$. Then the restriction $\mathcal{Q}$ of $\mathcal{P}$ to $\Delta^{(2)}$ is such that $(N^\Delta, F^\Delta, \Delta, \mathcal{Q})$ is a $k$-TOD, and $E^{\mathcal{P}}$ is permutationally isomorphic to $F^{\mathcal{Q}}$. Further, if in addition $(M, G, \Omega, \mathcal{P})$ is symmetric, then also $(N^\Delta, F^\Delta, \Delta, \mathcal{Q})$ is symmetric.*

*Proof.* Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$; so $\mathcal{P}(\omega) = \{P_1(\omega), P_2(\omega), \ldots, P_k(\omega)\}$. Since $\mathcal{P}$ is refined by $\mathrm{Orbl}(M, \Omega)$, each of the $P_i(\omega)$ is $M_\omega$-invariant and hence also $N_\omega$-invariant. Thus each $Q_i(\omega) := \Delta \cap P_i(\omega)$ is $N_\omega$-invariant. Define $\mathcal{Q}(\omega) = \{Q_i(\omega) \mid 1 \leq i \leq k\}$. Observe that $\bigcup_i Q_i(\omega) = \Delta \setminus \{\omega\}$, and if $i \neq j$, then $Q_i(\omega) \cap Q_j(\omega) = \emptyset$; so $\mathcal{Q}(\omega)$ is a partition of $\Delta \setminus \{\omega\}$. Also, $\mathcal{Q}(\omega)$ is invariant under $N_\omega$.

By assumption $E^{\mathcal{P}}$ is transitive, and since $E \leq G_\omega$, $E$ also acts transitively on $\mathcal{P}(\omega)$. Further, since $E$ normalises $N$ and fixes $\omega$, it follows that $E$ fixes $\Delta$ setwise. Thus, $E$ has an induced action on $Q(\omega)$ given by $Q_i(\omega)^g = \Delta \cap P_i(\omega)^g$ ($g \in E$, $i \leq k$), which is permutationally isomorphic to its actions on $\mathcal{P}(\omega)$ and $\mathcal{P}$. In particular, $E$ acts transitively on $\mathcal{Q}(\omega)$, and all the $Q_i(\omega)$ are nonempty. Note that $F_\omega = N_\omega E$ and the induced action on $\mathcal{Q}(\omega)$ satisfies $F_\omega^{\mathcal{Q}(\omega)} = (N_\omega E)^{\mathcal{Q}(\omega)} = E^{\mathcal{Q}(\omega)}$. By Lemma 2.3, the corresponding partition $\mathcal{Q}$ of $\Omega^{(2)}$ is $F$-invariant, and the $F$-action on $\mathcal{Q}$ is equivalent to the $F_\omega$-action on $\mathcal{Q}(\omega)$. Thus $F^{\mathcal{Q}}$ is permutationally isomorphic to $E^{\mathcal{P}}$, and $(N^\Delta, F^\Delta, \Delta, \mathcal{Q})$ is a $k$-TOD. From the definition of $Q$ (preceding Lemma 2.3) it is clear that $Q_i \subseteq P_i$ for each $i$, and therefore $Q_i = P_i \cap (\Delta \times \Delta)$, for $1 \leq i \leq k$.

Assume in addition that $(M, G, \Omega, \mathcal{P})$ is symmetric, that is, each $P_i$ is symmetric. Let $O \in \mathrm{Orbl}(M, \Omega)$, and let $\hat{O} \in \mathrm{Orbl}(N, \Delta)$ be such that $\hat{O}(\omega) \subseteq O(\omega)$. Then $\hat{O}^*(\omega) \subseteq O^*(\omega)$. Assume that $O(\omega) \subseteq P_i(\omega)$. Then, since $P_i$ is symmetric, $O(\omega) \cup O^*(\omega) \subseteq P_i(\omega)$. By the definition of $Q_i(\omega)$, $\hat{O}(\omega) \cup \hat{O}^*(\omega) \subseteq Q_i(\omega)$. It follows that $Q_i$ is symmetric, and so $(N^\Delta, F^\Delta, \Delta, \mathcal{Q})$ is symmetric. □

Our final construction is not elementary, since it relies on an application of the result of Fein, Kantor and Schacher [6] that a transitive permutation group contains a fixed-point-free element of prime-power order. This result relies on the finite simple group classification. Recall that we may always assume, for a TOD $(M, G, \Omega, \mathcal{P})$, that $M$ is normal in $G$ and hence that $G$ leaves $\mathrm{Orbl}(M, \Omega)$ invariant.

**Theorem 3.6.** *If $(M, G, \Omega, \mathcal{P})$ is a $k$-TOD with $M$ normal in $G$, then there exists a $p$-TOD $(M, H, \Omega, \mathcal{Q})$ for some prime divisor $p$ of $k$ and some partition $\mathcal{Q}$ of $\Omega^{(2)}$ refined by $\mathcal{P}$, where $H = \langle M, \tau\rangle$ for some $\tau \in G_\omega \setminus M_\omega$ where $\omega \in \Omega$. In particular, $\tau$ fixes no element of* $\mathrm{Orbl}(M, \Omega)$.

*Proof.* Let $(M, G, \Omega, \mathcal{P})$ be a $k$-TOD with $M$ normal in $G$. Let $\omega \in \Omega$. Since $M$ is transitive, $G = MG_\omega$; so $G^{\mathcal{P}} = G_\omega^{\mathcal{P}}$. Thus $G_\omega^{\mathcal{P}}$ is transitive, and it follows from [6] that, for some prime $p$, $G_\omega$ contains an element $\tau$ of $p$-power order such that $\tau^{\mathcal{P}}$ has no fixed points in $\mathcal{P}$. Label the parts of $\mathcal{P}$ as $P_{ij}$, so that the $i^{\text{th}}$-orbit of $\langle \tau^{\mathcal{P}}\rangle$ in $\mathcal{P}$ is $\{P_{i,j} \mid 1 \le j \le p^{a_i}\}$, $a_i \ge 1$, and $P_{ij}^\tau = P_{i,j+1}$ (reading the second subscript modulo $p^{a_i}$). For $l = 1, \ldots, p$, define $Q_l = \bigcup_i (P_{i,l} \cup P_{i,l+p} \cup \cdots \cup P_{i,l-p+p^{a_i}})$. Then $\mathcal{Q} = \{Q_1, \ldots, Q_p\}$ is permuted cyclically by $\tau$, and $\mathcal{Q}$ is a partition of $\Omega^{(2)}$ refined by $\mathcal{P}$. Thus $(M, \langle M, \tau\rangle, \Omega, \mathcal{Q})$ is a $p$-TOD. Since $\mathcal{Q}$ is refined by $\mathcal{P}$, it is also refined by $\mathrm{Orbl}(M, \Omega)$, and it follows that $\tau$ acts on $\mathrm{Orbl}(M, \Omega)$ with no fixed points. □

This result has an immediate consequence:

**Lemma 3.7.** *Let $(M, G, \Omega, \mathcal{P})$ be a $k$-TOD such that $G$ has a regular normal subgroup $N$ contained in $M$. Then $N$ is soluble.*

*Proof.* By Lemma 2.2, $(N, G, \Omega, \mathcal{P})$ is a $k$-TOD, and then, by Theorem 3.6, there exists an element $\tau \in G \setminus N$ that fixes no element of $\mathrm{Orbl}(N, \Omega)$. Since $N$ is regular on $\Omega$, it follows that $\tau$ fixes no non-identity element of $N$. Thus the automorphism of $N$ induced by conjugation by $\tau$ is fixed-point-free, and hence $N$ is soluble, see [9, Thm. 1.48]. □

## 4. TODs and imprimitive group actions

4.1. **TODs on blocks of imprimitivity.** We show that the induced configuration of a $k$-TOD $(M, G, \Omega, \mathcal{P})$ on a block $B$ of imprimitivity for $G$ on $\Omega$ is also a $k$-TOD. Let $(M, G, \Omega, \mathcal{P})$ be a $k$-TOD, and let $\mathcal{B}$ be a $G$-invariant partition of $\Omega$. Let $\mathcal{P}_B := \{P_1^B, P_2^B, \ldots, P_k^B\}$, where $P_i^B := P_i \cap (B \times B)$. Then each $P_i^B$ is a union of $M_B$-orbitals on $B$, and $\mathcal{P}_B$ is a partition of $\mathcal{B}^{(2)}$. We denote the setwise stabilisers of $B$ in $M$, $G$ by $M_B$ and $G_B$, respectively.

**Lemma 4.1.** *Let $(M, G, \Omega, \mathcal{P})$ be a $k$-TOD with $M$ normal in $G$. Then for a non-trivial block $B$ of imprimitivity for $G^\Omega$, $(M_B^B, G_B^B, B, \mathcal{P}_B)$ is a $k$-TOD; further, $G^{\mathcal{P}}$ is permutationally isomorphic to $G_B^{\mathcal{P}_B}$.*

*Proof.* The setwise stabiliser $M_B$ has no fixed points in $\Omega$ and is normalised by $G_\omega$. Also, since $M^{\mathcal{P}} = 1$, we have that $G^{\mathcal{P}} = G_\omega^{\mathcal{P}}$ is transitive. The result now follows from Lemma 3.5 applied with $N = M_B$, $E = G_\omega$, and $\Delta = B$. □

By choosing $B$ to be a minimal block of imprimitivity, we may assume that the group $G_B^B$ is primitive. This suggests studying $k$-TODs $(M, G, \Omega, \mathcal{P})$ with $G$ primitive on $\Omega$.

**Proposition 4.2.** *Let* $(M, G, \Omega, \mathcal{P})$ *be a* $k$*-TOD such that* $G$ *is primitive on* $\Omega$*. Then* $G^\Omega$ *is of O'Nan-Scott type* HA , AS, SD, CD *or* PA (*as defined in* [22]).

*Proof.* This is an immediate consequence of Lemma 3.7. □

Such TODs are investigated further in [10] and, in particular, a classification is obtained of cyclic $p^a$-TODs where $p$ is a prime. It is shown there, in particular, that there exist TODs corresponding to each of the five O'Nan-Scott types specified in the proposition.

4.2. **Quotients of TODs.** Let $(M, G, \Omega, \mathcal{P})$ be a TOD and let $\mathcal{B}$ be a $G$-invariant partition of $\Omega$. There is a natural map from $\Omega^{(2)}$ to $\mathcal{B}\times\mathcal{B}$ given by $(\omega, \omega') \to (B, B')$, where $\omega \in B \in \mathcal{B}$ and $\omega' \in B' \in \mathcal{B}$. This induces a map from subsets of $\Omega^{(2)}$ to subsets of $\mathcal{B}\times\mathcal{B}$. However, there are at least two reasons why, in general, a partition $\mathcal{P}$ of $\Omega^{(2)}$ is not mapped to a partition of $\mathcal{B}^{(2)}$. First there is the problem that distinct points $\omega, \omega'$ may lie in the same block of $\mathcal{B}$. One might hope still to achieve a partition of $\mathcal{B}^{(2)}$ simply by ignoring such pairs. However, the second problem is that disjoint subsets of $\Omega^{(2)}$ may correspond to non-disjoint subsets of $\mathcal{B}^{(2)}$. This second problem makes it impossible, in general, to find a natural partition of $\mathcal{B}^{(2)}$corresponding to a given partition of $\Omega^{(2)}$. In particular, there seems to be no natural construction of a TOD for the actions of $M$ and $G$ on $\mathcal{B}$ from a given TOD $(M, G, \Omega, \mathcal{P})$. This is demonstrated by the following simple example.

**Example 4.3.** Let $M = \langle(123456789)\rangle \cong \mathbb{Z}_9$, $G = \mathrm{D}_{18}$, and $\Omega = \{1, 2, \ldots, 9\}$. Then $\mathrm{Orbl}(M, \Omega) = \{\Delta_i = (1, i)^M \mid 2 \le i \le 9\}$. Let $P_1 = \Delta_2 \cup \Delta_3 \cup \Delta_4 \cup \Delta_5$ and $P_2 = \Delta_6 \cup \Delta_7 \cup \Delta_8 \cup \Delta_9$. Then $\mathcal{P} = \{P_1, P_2\}$ is a partition of $\Omega^{(2)}$ and $(M, G, \Omega, \mathcal{P})$ is a 2-TOD. Now $B_1 = \{1, 4, 7\}$, $B_2 = \{2, 5, 8\}$ and $B_3 = \{3, 6, 9\}$ form a $G$-invariant partition $\mathcal{B}$ of $\Omega$, but the images of both $P_1$ and $P_2$, under the map $\Omega^{(2)} \to \mathcal{B} \times \mathcal{B}$ defined above, are $\mathcal{B} \times \mathcal{B}$.

However, we show in the next section that, for a cyclic TOD $(M, G, \Omega, \mathcal{P})$ with $G$-invariant partition $\mathcal{B}$ of $\Omega$, it is possible to construct an induced quotient TOD on $\mathcal{B}$ of the same index. (This is Theorem 1.2, stated in the introduction.)

## 5. Cyclic TODs

To prove Theorem 1.2, we first prove a very useful result about cyclic TODs. A similar result can be found in [11].

**Proposition 5.1.** *Let* $(M, G, \Omega, \mathcal{P})$ *be a cyclic* $k$*-TOD and let* $K$ *be the kernel of the action of* $G$ *on* $\mathcal{P}$*. Then each element* $\tau \in G \setminus K$ *has exactly one fixed point in* $\Omega$*.*

*Proof.* Let $\omega \in \Omega$. We have $G = KG_\omega$, and so $G = \langle K, \sigma\rangle$ for some $\sigma \in G_\omega$. Let $k = \prod_{i=1}^r p_i^{e_i}$ for distinct primes $p_i$, $e_i \ge 1$, and $r \ge 1$. Let $\tau \in G \setminus K$. Then there exists $i$ such that the order of $\tau$ modulo $K$ is divisible by $p_i$, and hence $\tau \notin \langle K, \sigma^{p_i^{e_i}}\rangle$. Let $\hat{\mathcal{P}}$ be the partition of $\Omega^{(2)}$ such that each part is the union of the parts of $\mathcal{P}$ contained in some orbit of $\langle(\sigma^{p_i^{e_i}})^{\mathcal{P}}\rangle$. Then by Lemma 3.4 (1), $(M, G, \Omega, \hat{\mathcal{P}})$ is a cyclic $p_i^{e_i}$-TOD. Set $\hat{M} = \langle K, \sigma^{p_i^{e_i}}\rangle$. Then by Lemma 2.2, $(\hat{M}, G, \Omega, \hat{\mathcal{P}})$ is a cyclic

$p_i^{e_i}$-TOD, and $\hat{M}$ is the kernel of the action of $G$ on $\hat{\mathcal{P}}$. There is a $p_i$-element $\sigma'$ such that $G = \langle \hat{M}, \sigma' \rangle = \langle \hat{M}, \sigma \rangle$ (taking $\sigma'$ to be the "$p_i$-part" of $\sigma$).

Then $\tau = x(\sigma')^l$ for some $x \in \hat{M}$ and some integer $l$ not divisible by $p_i^{e_i}$. Since $x$ fixes $\hat{\mathcal{P}}$ pointwise, the $\langle \tau \rangle$-action on $\hat{\mathcal{P}}$ is equivalent to the action of $\langle \sigma^l \rangle$ on $\hat{\mathcal{P}}$. Thus $\langle \tau \rangle$ is nontrivial and half-transitive on $\hat{\mathcal{P}}$. In particular, $\tau$ fixes no element of $\hat{\mathcal{P}}$, and so $\tau$ fixes no $(\omega, \omega') \in \Omega^{(2)}$ for any distinct $\omega, \omega' \in \Omega$. Hence $\tau$ fixes at most one point of $\Omega$.

If $\tau$ has $p_i$-power order, then, since $p_i \nmid |\Omega|$ by Lemma 2.5, $\tau$ fixes at least one point of $\Omega$, so that it fixes exactly one point of $\Omega$.

Suppose now that the order $o(\tau) = n_1 n_2$ is such that $n_1$ is a $p_i$-power, $\gcd(n_1, n_2) = 1$, and $n_2 > 1$, and write $\tau = \tau_1 \tau_2$ such that $o(\tau_i) = n_i$ and $\tau_1\tau_2 = \tau_2\tau_1$. Since $|G : \hat{M}| = p_i^{e_i}$, it follows that $\tau_2 \in \hat{M}$, and therefore $\tau_1$ is of $p_i$-power order and lies in $G \backslash \hat{M}$. By the argument of the previous paragraph, $\tau_1$ has exactly one fixed point in $\Omega$. Let $\Delta_1, \Delta_2, \ldots, \Delta_t$ be the $\langle \tau \rangle$-orbits in $\Omega$. Since $\langle \tau_1 \rangle$ is a normal subgroup of $\langle \tau \rangle$, we have that $\langle \tau_1 \rangle$ acts on each $\Delta_j$ half-transitively. Thus either $\langle \tau_1 \rangle$ acts on $\Delta_j$ trivially, or $|\Delta_j|$ is divisible by $p_i$. Since $\tau_1$ has exactly one fixed point in $\Omega$, it follows that exactly one of the $\Delta_j$ has size 1 and all the others have size a multiple of $p_i$. Therefore, $\tau$ fixes exactly one point in $\Omega$. □

Now we deduce a corollary for imprimitive cyclic TODs.

**Lemma 5.2.** *Let $(M, G, \Omega, \mathcal{P})$ be a cyclic $k$-TOD. Let $K$ be the kernel of the $G$-action on $\mathcal{P}$, and let $\mathcal{B}$ be a nontrivial $G$-invariant partition of $\Omega$. Then each element of $G \setminus K$ fixes exactly one block of $\mathcal{B}$.*

*Proof.* Here $G = \langle K, \sigma \rangle$ for some $\sigma \in G$ such that $\sigma^k \in K$ and $G/K \cong \mathbb{Z}_k$. Then, by Lemma 2.2, $(K, G, \Omega, \mathcal{P})$ is a cyclic $k$-TOD. Write $\mathcal{B} = \{B_0, B_1, \ldots, B_t\}$ for some $t \geq 2$. Let $\tau \in G \setminus K$. By Proposition 5.1, $\tau$ fixes a point of $\Omega$ and hence fixes setwise a block of $\mathcal{B}$, say $B_0$. Now $\tau = f\sigma^r$ for some integer $r$ and $f \in K$, and $\tau$ has order $k_0$ modulo $K$ for some $k_0 | k$ with $k_0 > 1$. Since $K$ acts trivially on $\mathcal{P}$, $\langle \tau^{\mathcal{P}} \rangle = \langle (\sigma^r)^{\mathcal{P}} \rangle$ has $k/k_0$ orbits of length $k_0$ in $\mathcal{P}$. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$, and observe that, for any $i \neq 0$,

$$(P_1 \cap (B_0 \times B_i)) \cup \cdots \cup (P_k \cap (B_0 \times B_i)) = B_0 \times B_i, \text{ and}$$
$$(P_j \cap (B_0 \times B_i)) \cap (P_{j'} \cap (B_0 \times B_i)) = \emptyset \quad \text{if } j \neq j',$$

where $B_0 \times B_i = \{(\omega, \omega') \mid \omega \in B_0,\ \omega' \in B_i\}$. Suppose that $\tau$ fixes $B_i$ setwise for some $i \in \{1, 2, \ldots, t\}$. Then $(B_0 \times B_i)^\tau = B_0 \times B_i$, and for any $j \in \{1, 2, \ldots, k\}$, there exists $j' \in \{1, 2, \ldots, k\}$ such that

$$(P_j \cap (B_0 \times B_i))^\tau = P_{j'} \cap (B_0 \times B_i).$$

If $P_j \cap (B_0 \times B_i) \neq \emptyset$, then for each of the $k_0$ parts $P_{j'}$ in the $\langle \tau^{\mathcal{P}} \rangle$-orbit containing $P_j$, we have $|P_j \cap (B_0 \times B_i)| = |P_{j'} \cap (B_0 \times B_i)|$. Since this is true for all $\langle \tau^{\mathcal{P}} \rangle$-orbits, it follows that $k_0$ divides $|B_0 \times B_i|$. However, by Lemma 4.1, $(K_{B_0}, G_{B_0}^{B_0}, B_0, \mathcal{P}_{B_0})$ is a $k$-TOD, and hence by Lemma 2.5, $|B_0| \equiv 1 \pmod{k}$. Therefore, $|B_0 \times B_i| = |B_0|^2 \equiv 1 \pmod{k}$, and so $|B_0 \times B_i| \equiv 1 \pmod{k_0}$, which is a contradiction since $k_0 > 1$. Therefore, $B_0$ is the unique fixed block of $\tau$. □

We are now ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let $K$ be the kernel of $G$ acting on $\mathcal{P}$. Then $G/K \cong \mathbb{Z}_k$, and $G = \langle K, \sigma \rangle$ for some $\sigma \in G$ such that $\sigma$ normalises $K$ and $\sigma^k \in K$. Since $K$

is transitive on $\Omega$, we have that $G = KG_\omega$, where $\omega \in \Omega$. Hence $\sigma = f\sigma'$ where $f \in K$ and $\sigma' \in G_\omega$. Since $f$ fixes every element of $\mathcal{P}$, $\langle\sigma'\rangle$ induces a transitive action on $\mathcal{P}$. Thus, without loss of generality, we may assume that $\sigma \in G_\omega$. Let $\mathcal{B} = \{B_0, B_1, \ldots, B_t\}$ be a $G$-invariant partition of $\Omega$ such that $\omega \in B_0$. Then in particular $B_0^\sigma = B_0$, and hence $\sigma$ normalises $K_{B_0}$. Let $\mathcal{D}$ be the set of $K_{B_0}$-orbits in $\Omega \setminus B_0$. Then $\mathcal{D}$ is $\langle\sigma\rangle$-invariant.

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ be such that $P_i^\sigma = P_{i+1}$ for each $i < k$ and $P_k^\sigma = P_1$, and let $\mathcal{P}(B_0) = \{P_1(B_0), P_2(B_0), \ldots, P_k(B_0)\}$, where

$$P_i(B_0) = \{\alpha \in \Omega \setminus B_0 \mid (\beta, \alpha) \in P_i, \text{ for some } \beta \in B_0\}.$$

Then

$$P_i(B_0)^\sigma = \{\alpha^\sigma \in \Omega \setminus B_0 \mid (\beta^\sigma, \alpha^\sigma) \in P_i^\sigma, \text{ for some } \beta^\sigma \in B_0\} = P_{i+1}(B_0),$$

reading the subscripts modulo $k$. Since $K_{B_0}$ acts trivially on $\mathcal{P}$, it follows that $K_{B_0}$ fixes each $P_i(B_0)$ setwise, and hence each $P_i(B)$ is a union of some subset of $\mathcal{D}$.

Suppose that $\Delta \in \mathcal{D}$ is contained in $P_1(B_0)$, and suppose that $B \in \mathcal{B} \setminus \{B_0\}$ is such that $B \cap \Delta$ contains a point $\alpha$. Suppose further that $1 \le i \le k$, and $B \cap \Delta^{\sigma^i}$ also contains a point, say $\beta$. Then $\beta = \delta^{\sigma^i}$ for some $\delta \in \Delta$, and since $K_{B_0}$ is transitive on $\Delta$, $\delta = \alpha^g$ for some $g \in K_{B_0}$. Thus $\beta = \alpha^{g\sigma^i} \in B \cap B^{g\sigma^i}$, and hence $g\sigma^i$ fixes $B$. However, $g\sigma^i \in G_{B_0}$ and $B \ne B_0$. It follows from Lemma 5.2 that $g\sigma^i \in K$. Hence $\sigma^i \in K$, and so $i = k$. Thus the $k$ sets $\Delta, \Delta^\sigma, \ldots, \Delta^{\sigma^{k-1}} \in \mathcal{D}$ meet disjoint subsets of $\mathcal{B} \setminus \{B_0\}$. Moreover, $\Delta^{\sigma^i} \subseteq (P_1(B_0))^{\sigma^i} = P_{i+1}(B_0)$ for each $i < k$.

For a $K_{B_0}$-orbit $\Delta$, let $\mathcal{B}(\Delta)$ denote the subset of blocks $B$ of $\mathcal{B} \setminus \{B_0\}$ such that $B \cap \Delta \ne \emptyset$. Suppose that $\mathcal{B}(\Delta) \cap \mathcal{B}(\Delta')$ (where $\Delta, \Delta' \in \mathcal{D}$) contains a block $B$, and let $B'$ be an arbitrary block in $\mathcal{B}(\Delta)$. Then, since $B \cap \Delta \ne \emptyset$, $B' \cap \Delta \ne \emptyset$, and $\Delta$ is a $K_{B_0}$-orbit, some element $x \in K_{B_0}$ maps a point of $B \cap \Delta$ to a point of $B' \cap \Delta$, and hence $B^x = B'$. Since $\Delta'$ is a $K_{B_0}$-orbit, we have $(B \cap \Delta')^x = B' \cap \Delta'$, and therefore $B' \in \mathcal{B}(\Delta')$. Thus $\mathcal{B}(\Delta) \subseteq \mathcal{B}(\Delta')$, and a similar argument proves that $\mathcal{B}(\Delta') \subseteq \mathcal{B}(\Delta)$. Thus, for $\Delta, \Delta' \in \mathcal{D}$, $\mathcal{B}(\Delta)$ and $\mathcal{B}(\Delta')$ are either equal or disjoint. It may happen that $\mathcal{B}(\Delta) = \mathcal{B}(\Delta')$ for distinct $K_{B_0}$-orbits $\Delta, \Delta'$, but we have just proved that, in this case, $\Delta$ and $\Delta'$ lie in different $\langle\sigma\rangle$-orbits. Thus $\langle\sigma\rangle$ permutes the set $\{\mathcal{B}(\Delta) \mid \Delta \in \mathcal{D}\}$ with all orbits of length $k$. Suppose that $\langle\sigma\rangle$ has $m$ orbits in this set, and suppose, without loss of generality, that $\mathcal{B}(\Delta_1), \ldots, \mathcal{B}(\Delta_m)$ are representatives of these $m$ orbits. Define $Q_1(B_0) := \mathcal{B}(\Delta_1) \cup \cdots \cup \mathcal{B}(\Delta_m)$ and, for $2 \le i \le k$, set $Q_i(B_0) := (Q_1(B_0))^{\sigma^{i-1}}$, and set $\mathcal{Q}(B_0) := \{Q_i(B_0) \mid 1 \le i \le k\}$. Then $\langle\sigma\rangle$ is transitive on $\mathcal{Q}(B_0)$, and it follows from Lemma 2.3 that $(M^\mathcal{B}, G^\mathcal{B}, \mathcal{B}, \mathcal{Q})$ is a cyclic $k$-TOD, where $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ with $Q_i = \{(B_0, B)^g \mid B \in Q_i(B_0), g \in G\}$ (as defined before Lemma 2.3). □

## 6. Explicit construction for cyclic TODs

For a cyclic $k$-TOD $(M, G, \Omega, \mathcal{P})$, we may assume by Lemma 2.2 that $M$ is normal in $G$ and therefore that there exists $\sigma \in G \setminus M$ such that $\sigma$ normalises $M$ and $\langle\sigma\rangle$ acts transitively on $\mathcal{P}$. The following is a consequence of Proposition 3.3, and gives a criterion for a transitive permutation group $M$ to give rise to a cyclic TOD.

**Lemma 6.1.** *Let $M$ be a transitive permutation group on $\Omega$, and let*

$$\sigma \in \mathbf{N}_{\mathrm{Sym}(\Omega)}(M) \quad \textit{and} \quad G = \langle M, \sigma \rangle < \mathrm{Sym}(\Omega).$$

*Then there exists a partition $\mathcal{P}$ of $\Omega^{(2)}$ such that*

(i) *$(M, G, \Omega, \mathcal{P})$ is a $k$-TOD if and only if $k$ divides the size of each $\langle \sigma \rangle$-orbit on $\mathrm{Orbl}(M, \Omega)$;*

(ii) *$(M, G, \Omega, \mathcal{P})$ is a symmetric $k$-TOD if and only if $k$ divides the size of each $\langle \sigma \rangle$-orbit on $\mathrm{Orbl}(M, \Omega)$, and for each $\Delta \in \mathrm{Orbl}(M, \Omega)$ and each $\tau \in \langle \sigma \rangle \setminus \langle \sigma^k \rangle$, $\Delta^\tau \neq \Delta^*$, where $\Delta^*$ is the paired orbital of $\Delta$.*

*Proof.* Part (i) follows immediately from Proposition 3.3. For part (ii), the extra condition is that, for each symmetric $\langle \sigma \rangle$-orbit $Q$ in $\mathrm{Orbl}(M, \Omega)$, there exists a symmetric $k$-part partition $\mathcal{B}(Q)$ with appropriate $\langle \sigma \rangle$-action. Now $Q$ is a symmetric $\langle \sigma \rangle$-orbit if and only if $Q = Q^*$, that is, $\Delta^* \in Q$ whenever $\Delta \in Q$. We require that each part $B \in \mathcal{B}(Q)$ should be symmetric, that is, $\Delta^* \in B$ whenever $\Delta \in B$. Since $\mathcal{B}(Q) = \{B^{\sigma^i} \mid 0 \leq i < k\}$, this condition implies that, for each $\Delta \in \mathrm{Orbl}(M, \Omega)$, $\Delta^{\sigma^i} \neq \Delta^*$ for $i = 1, \ldots, k-1$ (either because $\Delta$ and $\Delta^*$ lie in different $\langle \sigma \rangle$-orbits, or because they lie in the same block $B$ of $\mathcal{B}(Q)$, where $\Delta, \Delta^* \in Q$). Conversely, the condition $\Delta^{\sigma^i} \neq \Delta^*$ for all $\Delta$ and for $i = 1, \ldots, k-1$ enables us to define a $G$-invariant partition with all parts symmetric, for each symmetric $G$-orbit $Q$. (Take $B = \{\Delta^{\sigma^{ki}} \mid i \geq 0\}$.) □

If an element $\sigma \in \mathbf{N}_{\mathrm{Sym}(\Omega)}(M)$ has $p$-power order with $p$ prime, and $\sigma$ fixes no element of $\mathrm{Orbl}(M, \Omega)$, then $p$ divides the size of every $\langle \sigma \rangle$-orbit on $\mathrm{Orbl}(M, \Omega)$. Suppose that $\tau$ is an automorphism of $M$ that normalises a point stabilizer in $M$. Then $\tau$ is induced by some element $\tilde{\tau} \in \mathrm{Sym}(\Omega)$ such that $\tilde{\tau} \in \mathbf{N}_{\mathrm{Sym}(\Omega)}(M)$. Thus $\tilde{\tau}$ acts on $\mathrm{Orbl}(M, \Omega)$. By Lemma 6.1, we have a criterion for a transitive permutation group to have a TOD of prime index in terms of certain special automorphisms of the group.

**Corollary 6.2.** *A transitive group $M$ acting on $\Omega$ has a TOD of prime index $p$ if and only if $M$ has an automorphism $\tau$ of $p$-power order such that $\tau$ normalises some point stabiliser in $M$, and $\tilde{\tau}$ fixes no element of $\mathrm{Orbl}(M, \Omega)$, where $\tilde{\tau}$ is as above.*

To conclude this section, we construct explicit examples of cyclic TODs for all values of $n$ and $k$ occurring in Theorem 1.1.

**Construction 6.3.** Let $n = r_1^{d_1} \ldots r_m^{d_m}$, where the $r_i$ are distinct primes, $d_i \geq 1$ and $m \geq 1$. Let $M = \mathbb{Z}_{r_1}^{d_1} \times \cdots \times \mathbb{Z}_{r_m}^{d_m}$. Then $\mathrm{Aut}\,(M) = \prod_{i=1}^m \mathrm{GL}(d_i, r_i) \geq \prod_{i=1}^m \mathrm{GL}(1, r_i^{d_i})$.

(a) Suppose that $r_i^{d_i} \equiv 1 \pmod{k}$ for each $i$. Choose $\sigma_i \in \mathrm{GL}(1, r_i^{d_i})$ such that $\sigma_i$ is of order $k$. Let $\sigma = \sigma_1 \ldots \sigma_m \in \mathrm{Aut}\,(M)$. Take $P_1(1)$ to consist of one representative of each of the $\langle \sigma \rangle$-orbits in $M \setminus \{1\}$, and set

$$P_1 = \{(x, y) \mid x, y \in M, xy^{-1} \in P_1(1)\}, \text{and}$$
$$\mathcal{P} = \{P_1^{\sigma^i} \mid 0 \leq i < k\}.$$

(b) Suppose that $r_i^{d_i} \equiv 1 \pmod{2k}$ whenever $r_i$ is odd, and $r_i^{d_i} \equiv 1 \pmod{k}$ if $r_i = 2$. For each $i$, if $r_i$ is odd, then let $\sigma_i \in \mathrm{GL}(1, r_i^{d_i})$ have order $2k$, and if $r_i = 2$, let $\sigma_i \in \mathrm{GL}(1, r_i^{d_i})$ have order $k$. Let $\sigma = \sigma_1\sigma_2 \dots \sigma_r \in \mathrm{Aut}\,(M)$. Take $P_1(1)$ to consist of one representative of each of the $\langle\sigma\rangle$-orbits in $M \setminus \{1\}$, and set

$$P_1 = \{(x, y) \mid x, y \in M, xy^{-1} \in P_1(1)\}, \text{and}$$
$$\mathcal{P} = \{P_1^{\sigma^i} \mid 0 \le i < k\}.$$

The next lemma shows that these constructions produce cyclic $k$-TODs.

**Lemma 6.4.** (i) *For $M, \sigma, \mathcal{P}, n, k$ as in Construction 6.3 (a), $(M, \langle M, \sigma\rangle, M, \mathcal{P})$ is a cyclic $k$-TOD of degree $n$; if in addition $k$ is odd, then $(M, \langle M, \sigma\rangle, M, \mathcal{P})$ is a symmetric cyclic $k$-TOD of degree $n$.*

(ii) *For $M, \sigma, \mathcal{P}, n, k$ as in Construction 6.3 (b), $(M, \langle M, \sigma\rangle, M, \mathcal{P})$ is a symmetric cyclic $k$-TOD of degree $n$.*

*Proof.* The cyclic group $\mathrm{GL}(1, r_i^{d_i})$ is regular on $\mathbb{Z}_{r_i}^{d_i} \setminus \{1\}$, and hence $\prod_{i=1}^m \mathrm{GL}(1, r_i^{d_i})$ is semiregular on $M \setminus \{1\}$. Thus in both constructions (given in Construction 6.3) $\langle\sigma\rangle$ acts semiregularly on $M \setminus \{1\}$, and so $\langle\sigma\rangle$ acts on $\mathrm{Orbl}(M, M)$ with all orbits of length $k$ in part (i) and length $2k$ or $k$ in part (ii). It follows from Lemma 6.1 that in both cases, $(M, \langle M, \sigma\rangle, M, \mathcal{P})$ is a cyclic $k$-TOD.

Now suppose that $M, \sigma, \mathcal{P}, n, k$ are as in Construction 6.3 (a) with $k$ odd, or as in Construction 6.3 (b). We show that the condition of Lemma 6.1 (ii) holds. Suppose to the contrary that $\Delta^{\sigma^i} = \Delta^*$, where $\Delta \in \mathrm{Orbl}(M, M)$ and $\sigma^i \notin \langle\sigma^k\rangle$. Then in particular $k \nmid i$. Now $\Delta(1) = \{x\}$ for some $x \in M \setminus \{1\}$, and $\Delta^*(1) = \{x^{-1}\}$. Since $\sigma^i$ fixes 1, $x^{\sigma^i} = x^{-1}$, and so $x^{\sigma^{2i}} = x$. Now all $\langle\sigma\rangle$-orbits in $M \setminus \{1\}$ have length $2k$ or $k$. Since $x^{\sigma^{2i}} = x$, we have that $k|2i$. Since $k \nmid i$, $k$ must be even. Thus the proof of part (i) is complete. Continuing with the proof of part (ii), by Lemma 2.5, $n$ is odd. It then follows from the definition of $\sigma$ that all $\langle\sigma\rangle$-orbits in $M \setminus \{1\}$ have length $2k$. Hence $2k|2i$ and $k|i$, which is a contradiction. So $\Delta^{\sigma^i} \ne \Delta^*$, satisfying Lemma 6.1 (ii). □

## 7. Degrees and indices

In this section, we prove a relation between index $k$ and degree $n$ for a cyclic $k$-TOD of degree $n$, and complete the proof of Theorem 1.1. First, we show that if $(M, G, \Omega, \mathcal{P})$ is a cyclic TOD, then some Sylow subgroups induce cyclic TODs. For a prime $r$, by $r^a \| n$ we mean that $r^a$ is the highest power of $r$ dividing $n$.

**Lemma 7.1.** *Let $(M, G, \Omega, \mathcal{P})$ be a cyclic $p^e$-TOD of degree $n$ with $M$ normal in $G$, where $p$ is a prime and $G = \langle M, \sigma\rangle$ for some element $\sigma$ of $p$-power order. Let $r$ be a prime such that $r^d \| n$ with $d > 0$, and let $R$ be a Sylow $r$-subgroup of $M$. Then there exist an element $\sigma_0 \in G$, an orbit $\Sigma$ of $R$ in $\Omega$, and a partition $\mathcal{Q}$ of $\Sigma^{(2)}$ such that $(R, \langle R, \sigma_0\rangle, \Sigma, \mathcal{Q})$ is a cyclic $p^e$-TOD of degree $r^d$. If in addition $(M, G, \Omega, \mathcal{P})$ is symmetric, then $(R, \langle R, \sigma_0\rangle, \Sigma, \mathcal{Q})$ is also symmetric.*

*Proof.* There exists $l \ge d$ such that $r^l = |R|$, so that $r^{l-d}$ is the order of a Sylow $r$-subgroup of $M_\omega$, where $\omega \in \Omega$. Let $\tau$ be an arbitrary element of $G \setminus \langle M, \sigma^p\rangle$. Then $\langle M, \sigma^p, \tau\rangle = G$. By Lemma 5.1, $\tau$ fixes a unique point in $\Omega$, and we denote the point by $\omega_\tau$. Thus $\tau \in G_{\omega_\tau}$, and it follows that $\tau$ normalises the point-stabilizer $M_{\omega_\tau}$.

Let $\rho \in G \setminus \langle M, \sigma^p\rangle$, and let $S$ be a Sylow $r$-subgroup of $M_{\omega_\rho}$. Then $|S| = r^{l-d}$, and $S^\rho$ is also a Sylow $r$-subgroup of $M_{\omega_\rho}$. By Sylow's theorem, $S^\rho = S^g$ for some $g \in M_{\omega_\rho}$. Thus $S^{\rho'} = S$, where $\rho' := \rho g^{-1} \in G \setminus \langle M, \sigma^p\rangle$. Since both $\rho$ and $g$ fix $\omega_\rho$, we have that $\rho'$ fixes $\omega_\rho$. Hence by Lemma 5.1, $\omega_{\rho'} = \omega_\rho$. Thus $S$ is an $r$-subgroup of $M$ for which there exists an element $\rho' \in G \setminus \langle M, \sigma^p \rangle$ such that $S^{\rho'} = S$ and $|S_{\omega_{\rho'}}| = r^{l-d}$.

Assume now that $X$ is maximal by inclusion among $r$-subgroups of $M$ such that there exists $\tau \in G \setminus \langle M, \sigma^p \rangle$ satisfying

$$X^\tau = X \quad \text{and} \quad |X_{\omega_\tau}| = r^{l-d}.$$

Let $N = \mathbf{N}_M(X)$, and let $Y$ be a Sylow $r$-subgroup of $N$. Note that $N$ has no fixed points in $\Omega$, for if $S$ is a Sylow $r$-subgroup of $M$ containing $X$, then $\mathbf{N}_S(X)$ properly contains $X$, and hence has no fixed points in $\Omega$. Now $\tau$ normalises $N$, $X \trianglelefteq Y$, and $Y^\tau$ is a Sylow $r$-subgroup of $N$. Thus $Y^\tau = Y^x$ for some $x \in N$, and so $Y^{\hat\tau} = Y$, where $\hat\tau := \tau x^{-1}$. Let $\Delta = \omega_\tau^N$, the orbit of $N$ containing $\omega_\tau$. Since $\tau$ fixes $\omega_\tau$ and normalises $N$, we have that $\tau$ fixes $\Delta$ setwise. Thus, in particular, $\Delta^{\hat\tau} = \Delta^{\tau x^{-1}} = \Delta$. By Lemma 3.5, $(N, \langle N, \tau \rangle, \Delta, \mathcal{P}')$ is a cyclic $p^e$-TOD for some partition $\mathcal{P}'$ of $\Delta^{(2)}$. Thus by Lemma 5.1, $\hat\tau$ fixes a point of $\Delta$, and so $\omega_{\hat\tau} \in \Delta$. Since $N$ is transitive on $\Delta$, we have $\omega_\tau^y = \omega_{\hat\tau}$ for some $y \in N$, and thus $X_{\omega_\tau}^y \le N_{\omega_\tau}^y = N_{\omega_{\hat\tau}}$. So $X_{\omega_\tau}^y \le X^y \cap N_{\omega_{\hat\tau}} = X \cap N_{\omega_{\hat\tau}} = X_{\omega_{\hat\tau}}$. Hence $r^{l-d} = |X_{\omega_\tau}| = |X_{\omega_\tau}^y| \le |X_{\omega_{\hat\tau}}| \le |Y_{\omega_{\hat\tau}}|$. However, since $r^{l-d} \| |M_{\omega_{\hat\tau}}|$, we have that $|Y_{\omega_{\hat\tau}}| = r^{l-d}$. Therefore, $Y$ is an $r$-subgroup of $M$ such that $Y^{\hat\tau} = Y$ and $|Y_{\omega_{\hat\tau}}| = r^{l-d}$. By the maximality of $X$, we have $Y = X$. Thus $X$ is a Sylow $r$-subgroup of $\mathbf{N}_M(X)$, and hence $X$ is a Sylow $r$-subgroup of $M$. In particular, $|X| = r^l$ and $X$ has no fixed points in $\Omega$.

By Sylow's theorem, $R = X^g$ for some $g \in M$. Let $\tau_0 = \tau^g$. Then by Lemma 3.5 applied to $R$ and $K = \langle R, \tau_0 \rangle$, there exists a $p^e$-TOD $(R, \langle R, \tau_0 \rangle, \Sigma, \mathcal{Q})$, where $\Sigma = \omega_{\tau_0}^R$ and $|\Sigma| = |R : R_{\omega_{\tau_0}}| = r^d$. If in addition $(M, \langle M, \sigma \rangle, \Omega, \mathcal{P})$ is symmetric, then by Lemma 3.5, $(R, \langle R, \tau_0 \rangle, \Sigma, \mathcal{Q})$ is symmetric. □

We now give a relation between the degrees and indices of cyclic $k$-TODs in the case where $k$ is a prime-power.

**Lemma 7.2.** *Let $p$ be a prime, and let $n = r_1^{d_1} r_2^{d_2} \dots r_m^{d_m}$ where the $r_i$ are distinct primes.*

(i) *If there exists a $p^e$-TOD of degree $n$, then $r_i^{d_i} \equiv 1 \pmod{p^e}$, for all $r_i$.*
(ii) *If there exists a symmetric $p^e$-TOD of degree $n$, then $r_i^{d_i} \equiv 1 \pmod{2p^e}$ for all odd $r_i$, and $r_i^{d_i} \equiv 1 \pmod{p^e}$ if $r_i = 2$.*

*Proof.* Let $(M, G, \Omega, \mathcal{P})$ be a cyclic $p^e$-TOD of degree $n$, where $G = \langle M, \sigma \rangle$ for some element $\sigma \in G$ of $p$-power order. Let $r$ be a prime such that $r^d \| n$ with $d > 0$. Let $R$ be a Sylow $r$-subgroup of $M$. By Lemma 7.1, there exists a $p^e$-TOD $(R, \langle R, \tau \rangle, \Sigma, \mathcal{Q})$ of degree $r^d$. Thus, by Lemma 2.5, $r^d \equiv 1 \pmod{p^e}$. If in addition $(M, \langle M, \sigma \rangle, \Omega, \mathcal{P})$ is symmetric, then by Lemma 7.1, $(R, \langle R, \tau \rangle, \Sigma, \mathcal{Q})$ is symmetric. Thus by Lemma 2.5, either $r^d \equiv 1 \pmod{2p^e}$, or $r = 2$ and $2^d \equiv 1 \pmod{p^e}$. □

We are now ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* If $k$ is a prime-power, then Theorem 1.1 follows from Lemmas 6.4 and 7.2. Thus we may assume that $k$ is not a prime-power. Let $(M, G, \Omega, \mathcal{P})$ be a cyclic $k$-TOD of degree $n$. Write $k = p_1^{e_1} p_2^{e_2} \dots p_l^{e_l}$, where $p_i$ are distinct primes, $e_i \ge 1$, and $l \ge 2$. Let $\sigma \in G \setminus M$ be such that $\langle \sigma \rangle$ is transitive on $\mathcal{P}$. Recall that we may take $\sigma \in G_\omega$. Let $k_i = k / p_i^{e_i}$, and let $\sigma_i = \sigma^{k_i}$. Then $\langle \sigma_i \rangle$ acts half-transitively on $\mathcal{P}$, and each $\langle\sigma_i\rangle$-orbit on $\mathcal{P}$ has size $p_i^{e_i}$. Hence $\langle\sigma_i\rangle$ acts on the set of $M$-orbitals such that each orbit has size divisible by $p_i^{e_i}$. By Lemma 6.1, there exists a cyclic $p_i^{e_i}$-TOD $(M, \langle M, \sigma_i\rangle, \Omega, \mathcal{P}_i)$ for some partition $\mathcal{P}_i$ of $\Omega^{(2)}$. By Lemma 7.2, if $r$ is a prime and $r^d \| |\Omega|$, then $r^d \equiv 1 \pmod{p_i^{e_i}}$. It then follows that $r^d \equiv 1 \pmod{k}$.

Assume further that $(M, G, \Omega, \mathcal{P})$ is symmetric. By Lemma 3.5 (taking $N = M$ and $E = \langle\sigma\rangle \le G_\omega$), $(M, \langle M, \sigma_i\rangle, \Omega, \mathcal{P}_i)$ is symmetric. By Lemma 7.2, if $r$ is odd and $r^d \| |\Omega|$, then $r^d \equiv 1 \pmod{2p_i^{e_i}}$. It then follows that $r^d \equiv 1 \pmod{2k}$ if $r$ is odd.

The converse assertion of Theorem 1.1 follows from Lemma 6.4. □

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